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Representations of non-commutative quantum mechanics and symmetries

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Abstract. We present a unified approach to representations of quantum mechanics on non-commutative spaces with general constant commutators of the phase-space variables. We find two phases and duality relations among them in arbitrary dimensions. Conditions for the physical equivalence of different representations of a given system are analyzed. Symmetries and classification of phase spaces are discussed. Especially, the dynamical symmetry of a physical system is investigated. Finally, we apply our analyses to the two-dimensional harmonic oscillator and the Landau problem.

1 Introduction

The problem of quantum mechanics on non-commutative spaces can be understood in the framework of deformation quantization. This is a subject with a long history starting with works of Wigner, Weyl and von Neumann (see [1] for a recent review). More recently, the investigation of non-commutative quantum mechanics was inspired by the development that led to non-commutative field theory. Namely, it was realized that low-energy effective field theory of various D-brane configurations has a configuration space which is described in terms of non-commuting, matrix-valued coordinate fields [2]. Then, it was shown that, in a certain limit, the entire string dynamics can be described by minimally coupled gauge theory on non-commutative space [3]. Intensive studies of field theories on various non-commutative spaces [4] were also inspired by the connection with M-theory compactifications [5] and more recently, by the matrix formulation of the quantum Hall effect [6]. In order to study phenomenological consequences of non-commutativity, a non-commutative deformation of the standard model has been constructed and analyzed [7].

In the last two years a lot of work has been done in analyzing and understanding quantum mechanics (QM) on non-commutative (NC) spaces [8–18] and also in applying it to different physical systems in order to test its relevance to the real world [19,20]. Still, there are many different views and approaches to non-commutative physics [21–23]. Some important questions, such as the physical equivalence of different non-commutative systems, as well as their relation to ordinary quantum mechanics with canonical variables have not been completely resolved. The symmetries and the physical content in different phases have not been completely elucidated, even in the simplest case of harmonic oscillator on the non-commutative plane.

In this paper, we present a unified approach to representations of NCQM in arbitrary dimensions. The conditions for the physical equivalence of different representations of a given system are analyzed. We show that there exist two phases in parameter space. Phase I can be viewed as a smooth deformation of ordinary QM, where all physical quantities have a smooth limit to physical quantities in ordinary QM. Phase II is qualitatively different from phase I and cannot be continuously connected to ordinary QM. There is a discrete duality transformation connecting the two phases.

Furthermore, we investigate symmetry transformations preserving commutators, the Hamiltonian and also the dynamical symmetry of the physical system. We analyze the angular momentum generators, and give conditions for their existence.

We demonstrate our general results on the simple example of a harmonic oscillator on a non-commutative plane. Especially, we describe the dynamical symmetry structure and discuss the uncertainty relations. Finally, we briefly comment on the NC Landau problem.

2 Non-commutative quantum mechanics and its representations

Let us start with the two-dimensional non-commutative coordinate plane X_1, X_2 and the corresponding momenta P_1, P_2, where X_i and P_i are hermitean operators. We describe a problem in four-dimensional phase space using variables $U = \{U_1, U_2, U_3, U_4\} = \{X_1, P_1, X_2, P_2\}$, where the U_i satisfy the general commutation relations

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$$[U_i, U_j] = \mathrm{i}M_{ij},\ i,j = 1,2,3,4, \tag{1}$$

and $M_{ij} = -M_{ji}$ are real constants (c-numbers). The antisymmetric matrix M is parametrized by six real parameters:

$$M = \begin{pmatrix} 0 & \hbar_1 & \theta & \phi_1 \\ -\hbar_1 & 0 & \phi_2 & B \\ -\theta & -\phi_2 & 0 & \hbar_2 \\ -\phi_1 & -B & -\hbar_2 & 0 \end{pmatrix}, \tag{2}$$

and the determinant $\det M = (\hbar_1\hbar_2 - \theta B + \phi_1\phi_2)^2$ is positive. The critical point $\det M = 0$ divides the space of the parameters into two phases: phase I for $\kappa = \hbar_1\hbar_2 - \theta B + \phi_1\phi_2 > 0$ and phase II for $\kappa < 0$. The ordinary, commutative space M_0,

$$M_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3}$$

has $\kappa = 1$ and belongs to phase I. Therefore, we can view phase I as a continuous, smooth deformation of ordinary quantum mechanics. The critical point $\kappa = 0$ corresponds to a reduction of the dimensions in phase space and to an infinite degeneracy of states and is related to the (noncommutative) Landau problem [9] (see also the "exotic" approach in [21]).

If we define the angular momentum J as

$$[J, X_a] = \mathrm{i}\varepsilon_{ab}X_b,\quad [J, P_a] = \mathrm{i}\varepsilon_{ab}P_b,\ a,b = 1,2, \tag{4}$$

or

$$[J, U_i] = \mathrm{i}E_{ij}U_j,\ i,j = 1,2,3,4, \tag{5}$$

then for a given regular matrix M we can construct the angular momentum only if $[E, M] = 0$. This condition is fulfilled when $\phi_1 = \phi_2$ and $\hbar_1 = \hbar_2$. Then

$$J = -\frac{1}{2}(EM^{-1})_{ij}U_iU_j. \tag{6}$$

We see that for general M the angular momentum in the usual sense may not exist. Moreover, even when it exists, it may have unusual properties. Namely, it was shown in [12] that in phase I a system could have an infinite number of states for a given value of the angular momentum, while in phase II the number of such states is finite.

Now, let us assume that the Hamiltonian of the system describes the motion of a single particle on a noncommutative plane:

$$H = \frac{1}{2}\mathbf{P}^2 + V(\mathbf{X}^2), \tag{7}$$

with a discrete spectrum E_{n_1,n_2}, where n_1, n_2 are nonnegative integers. The pair $(H(U), M)$ defines a system with a given energy spectrum and the corresponding energy eigenfunctions. We wish to characterize all systems $(H'(U'), M')$ with the same spectrum. The class of such systems is very large and can be defined by all real, nonlinear, regular transformations $U'_i = U'_i(U_j),\ U_i = U_i(U'_j)$. We restrict ourselves to linear transformations $Gl(4,\mathbb{R})$ in order to keep the matrix elements M'_{ij} independent of the phase-space variables. Among these, of special interest are the $O(4)$ orthogonal transformations changing the commutation relations, and the group of transformations isomorphic to $Sp(4)$ keeping M invariant. Systems with the same energy spectrum connected by transformations that keep the commutation relations invariant are physically equivalent. In both cases, the Hamiltonian generally changes, but the energy spectrum is invariant.

Let us consider $O(4)$ transformations. The important property [24,14] is that there exists an orthogonal transformation $\tilde{R}$ such that

$$\tilde{R}^{\mathrm{T}}M\tilde{R} = \begin{pmatrix} 0 & |\omega_1| & 0 & 0 \\ -|\omega_1| & 0 & 0 & 0 \\ 0 & 0 & 0 & |\omega_2| \\ 0 & 0 & -|\omega_2| & 0 \end{pmatrix}, \tag{8}$$

where $|\omega_1| \geq |\omega_2| \geq 0$ and $\det M = \omega_1^2\omega_2^2 \geq 0$. The matrix $\tilde{R}$ is unique up to the transformations $S \in SO(4)$:

$$\tilde{R}_s = S\tilde{R},\quad S^{\mathrm{T}}MS = M. \tag{9}$$

The first (second) phase is characterized by $\det\tilde{R} = +1$ ($\det\tilde{R} = -1$). For the two-dimensional case, the eigenvalues of the general matrix $\mathrm{i}M$, (2), are

$$\begin{aligned}\omega_{1,2} = &\frac{1}{2}\sqrt{(\theta - B)^2 + (\phi_1 + \phi_2)^2 + (\hbar_1 + \hbar_2)^2} \\ &\pm \frac{1}{2}\sqrt{(\theta + B)^2 + (\phi_1 - \phi_2)^2 + (\hbar_1 - \hbar_2)^2}.\end{aligned} \tag{10}$$

Notice that ω_1 is always positive, while ω_2 changes sign at the critical point $\det M = 0$, i.e., when $\theta B - \phi_1\phi_2 = \hbar_1\hbar_2$.

The matrix $\tilde{R}$ is universal, i.e., the following matrix exists: $R \in SO(4)$, with $\det R = 1$ such that

$$R^{\mathrm{T}}MR = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix} \equiv J_\omega, \tag{11}$$

regardless of ω_2 being positive, zero, or negative. When $\omega_2 < 0$, we use $\tilde{R} = RF$ to obtain (8), where the flip matrix $F \in O(4)$ is given by

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{12}$$

At the critical point ($\omega_2 = 0$) both R and RF satisfy (8).

The most general orthogonal matrix R depends on six continuous parameters. For fixed values $\{\omega_1, \omega_2\}$, the number of parameters of the matrix M is the same as the number of parameters of R. As we have already mentioned, there exist orthogonal matrices that commute with M, (9), and these matrices form a group isomorphic to $U(1) \times U(1)$. We can use this symmetry to fix two parameters in the matrix M, and we choose $\hbar_1 = \hbar_2 = 1$ or $\hbar_1 = -\hbar_2 = 1$. This parametrization covers all pairs $\{\omega_1, \omega_2\}$ such that $\omega_1^2 + \omega_2^2 \geq 2$.

The eigenvalues ω_1, ω_2 have the meaning of the "Planck" constants for the new variables:

$$\begin{aligned} &U_i^0 = R_{ik}^{\mathrm{T}} U_k, \\ &[X_a^0, P_b^0] = \mathrm{i}\omega_a \delta_{ab},\ [X_a^0, X_b^0] = [P_a^0, P_b^0] = 0. \end{aligned} \tag{13}$$

We have transformed the non-commutative system $(H(U), M)$ into $(H(RU^0), J_\omega)$ keeping the energy spectrum of the system invariant. Note that for the system $(H(RU^0),\ J_\omega)$ we cannot define the angular momentum. In order to connect a non-commutative system with a quantum mechanical system in ordinary space, we perform the following transformation:

$$U^0 = Du^0 = \begin{pmatrix} \sqrt{\omega_1} & 0 & 0 & 0 \\ 0 & \sqrt{\omega_1} & 0 & 0 \\ 0 & 0 & \sqrt{|\omega_2|} & 0 \\ 0 & 0 & 0 & \sqrt{|\omega_2|} \end{pmatrix} u^0, \tag{14}$$

where the variables $u^0 = \{u_1^0, u_2^0, u_3^0, u_4^0\}$ are canonical, i.e., $[x_a^0, p_b^0] = \mathrm{i}\delta_{ab}$ and $[x_a^0, x_b^0] = [p_a^0, p_b^0] = 0$. Now we have obtained $H(U) = H(RDu_0)$ with the standard canonical relations for M_0; see (3). The transformation D, see (14), is valid in both phases, but at the critical point it becomes singular. Also note that the composition $\mathcal{L}_0 = RD$ has a smooth limit when $M \to M_0$.

In order to make contact with other representations in the literature [9,15], we perform a symplectic transformation on the canonical variables u_i^0:

$$u_i = S_{ij} u_j^0 = V u_i^0 V^\dagger, \tag{15}$$

where S commutes with M_0 and

$$V = \exp\left(\mathrm{i} \sum v_{ij} u_i^0 u_j^0\right),\ \ VV^\dagger = 1,$$

is a unitary operator corresponding to the symplectic transformation S. This symplectic transformation generates a class of ordinary quantum mechanical systems which are physically (unitary) equivalent. Of course, the initial system $(H(U), M)$ is not physically equivalent to the canonical ones, but all corresponding physical quantities can be uniquely determined. In Fig. 1 we show a simple graphic description of the connection between different representations of NC quantum mechanics.

There is a "mirror-symmetric" diagram for phase II, obtained using the flip matrix F in (12), where $U' = R'FR^{\mathrm{T}}U$, $M' = R'FR^{\mathrm{T}}MRFR'^{\mathrm{T}}$, and $\mathcal{L}' = R'FR^{\mathrm{T}}\mathcal{L}F$. The matrix R' is any special orthogonal matrix. The universality of the matrix R means that we can choose R' and R to have the same functional dependence on the matrix elements M'_{ij} and M_{ij}, respectively. We have the

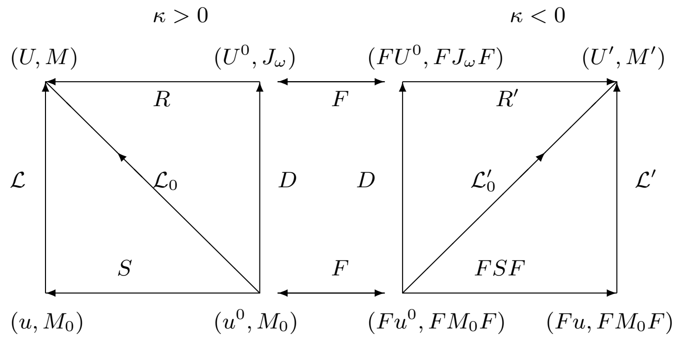

Fig. 1. Graphic representation of the transformations

discrete Z_2 symmetry connecting two components of the group $O(4)$, or more generally, $Gl(4, \mathbb{R})$.

Starting from the matrix M, we can construct the matrix R by finding eigenvalues and eigenvectors of the matrix $\mathrm{i}M$, i.e., $R = U_M U_J^\dagger$, where

$$U_M^\dagger(\mathrm{i}M)U_M = U_J^\dagger(\mathrm{i}J_\omega)U_J = \mathrm{diag}(\omega_1, -\omega_1, \omega_2, -\omega_2).$$

For example, for $\phi_1 = \phi_2 = 0$, we can write the matrix R in the following form:

$$R = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & \sin\varphi & 0 & \cos\varphi \\ 0 & \cos\varphi & 0 & -\sin\varphi \\ -\sin\varphi & 0 & \cos\varphi & 0 \end{pmatrix} \tag{16}$$

where we choose $\varphi \in (0, \pi/2)$, $\theta \geq 0$, $\theta + B \geq 0$ and

$$\begin{aligned} \cos\varphi &= \frac{1}{\sqrt{1 + (B + \omega_2)^2}} = \frac{\omega_2 + \theta}{\sqrt{1 + (\omega_2 + \theta)^2}} \\ &= \sqrt{\frac{\omega_1 - B}{\omega_1 + \omega_2}}. \end{aligned} \tag{17}$$

The basic relations are

$$\begin{aligned} \omega_1\omega_2 &= 1 - \theta B, \\ \omega_1 - \omega_2 &= \theta + B, \\ \omega_1 + \omega_2 &= \sqrt{(\theta - B)^2 + 4}. \end{aligned}$$

An interesting example of the matrix R is obtained in the case $\theta = B$, which corresponds to $\varphi = \pi/4$ in (16). In that case the matrix R does not depend on the non-commutativity parameters.

The R matrix was discussed in [14] in the context of the eigenvalue problem, but only in phase I. The authors of [14] stated that the matrix R became singular at the critical point. However, we wish to emphasize that the matrix R is a universal orthogonal matrix, valid in both phases and even at the critical point.

The transformations $\mathcal{L}$, S and $\mathcal{L}_0$ were discussed in [9, 15] for the case of a two-dimensional harmonic oscillator and parameterization $\hbar_1 = \hbar_2 = 1$ and $\phi_1 = \phi_2 = 0$, with the identification $u^0 = \{Q, P\}$ and $u = \{\alpha, \beta\}$. The authors of [9] treated the two phases separately, overlooking the universality of the transformation $\mathcal{L}_0 = RD$, whereas in [15] phase II was not analyzed.

We point out that the two systems $(H(U), M)$ and $(H'(U'), M')$ with the same energy spectrum and $M \neq M'$ are physically not equivalent. The condition for the physical equivalence is the same energy spectrum and the same commutation relations $M = M'$. Hence, even within the same phase two systems with the same energy spectrum can be quite different.

3 Two phases, duality and symmetries in arbitrary dimensions

The construction of different representations of quantum mechanics on a non-commutative plane can be easily generalized to arbitrary dimensions D. The regular, antisymmetric matrix M is parameterized by $D(2D-1)$ real parameters. We can classify non-commutative spaces according to $\{\omega_1, \omega_2, \ldots, \omega_D\}$, eigenvalues of the Hermitean matrix $\mathrm{i}M$. The determinant of the matrix M is positive, $\det M = \omega_1^2 \cdots \omega_D^2$. The critical point $\det M = 0$ divides the space of the parameters in two phases. In phase I, $\kappa = \omega_1 \cdots \omega_D > 0$, and in phase II, $\kappa < 0$. The critical point $\kappa = 0$ may have interesting physical applications, like the Landau problem in two dimensions.

In D dimensions, the angular momentum operators are generators of the coordinate space rotations:

$$[J_{ab}, X_c] = \mathrm{i}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})X_d,$$
$$[J_{ab}, P_c] = \mathrm{i}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})P_d,\ a, b, c, d = 1, \ldots, D,$$

and generally,

$$[J_{ab}, U_i] = (E_{ab})_{ij} U_j,\ i, j = 1, \ldots 2D.$$

For a regular matrix M we can construct the angular momentum generators $J_{ab} = -\frac{1}{2}(E_{ab}M^{-1})_{ij} U_i U_j$ only if $[E_{ab}, M] = 0$, for all $a, b = 1, \ldots, D$.

There are two sets of important transformations in the 2D phase space. One is a group of linear transformations $U_i' = S_{ij} U_j$ preserving M, $S^{\mathrm{T}} M S = M$. These transformations form a group $G(M)$ isomorphic to $Sp(2D)$. For every tranformation S there exists a unitary operator $V \sim \exp(\mathrm{i} \sum v_{ij} U_i U_j)$, and any two systems connected by such an S transformation are physically (unitary) equivalent.

The other important set of transformations are orthogonal transformations $O(2D)$ preserving the spectrum of the matrix ($\mathrm{i}M$), i.e., preserving $\omega_1, \ldots, \omega_D$ up to the signs. Transformations $R \in SO(2D)$ with $\det R = 1$ keep the system in the same phase. There is a discrete Z_2 transformation that changes the sign of one eigenvalue; we choose ω_D for definiteness. We represent this transformation using the flip matrix F, $F_{ii} = 1, i = 1, \ldots, 2(D-1)$, $F_{2D-1,2D} = F_{2D,2D-1} = 1$, and all other matrix elements zero. There is a simple example of this duality transformation that connects the two phases, obtained by choosing $R' = FRF$ (see Fig. 1):

$$\omega_D = -\omega_D',\ \omega_i = \omega_i', i = 1, \ldots, D-1,$$
$$FMF = M',\ \prod \omega_i = -\prod \omega_i'.$$

In general, duality is characterized by $|\omega_i| = |\omega_i'|, \forall i$ and $\kappa = -\kappa'$.

The matrix M can be brought to the J_ω form by the orthogonal transformation R; see (11). This $R \in SO(2D)$ matrix is unique up to the orthogonal transformations that preserve M. For fixed values $\{\omega_1, \ldots, \omega_D\}$, the number of parameters of the matrix M is the same as the number of parameters of R. The group of orthogonal transformations keeping M invariant, $SO(2D) \cap G(M)$, is isomorphic to $[U(1)]^D$ in the generic case. Using this freedom we can fix $M_{2i-1,2i} = 1, \forall i$ or we can put $M_{2D-1,2D} = -1$. So, using the symmetry we reduce the number of continuous phase-space parameters to $2D(D-1)$.

For a special choice of phase-space parameters M_{ij}, we can enlarge the symmetry group $[U(1)]^D$. The symmetries are characterized by degeneracy of eigenvalues $|\omega_i|$. If $k_1, \ldots, k_\alpha$ are frequencies of the appearance of $|\omega_1|, \ldots, |\omega_\alpha|$ in the spectrum of the matrix $\mathrm{i}M$, then the symmetry group $SO(2D) \cap G(M) \sim U(k_1) \times \cdots \times U(k_\alpha)$, where $\sum k = D$. Obviously, the largest symmetry group is $U(D)$. The sign of the product of eigenvalues determines the phase and the degeneracy among the $|\omega_i|$ determines the complete symmetry structure of the phase space. In this way, we classify the non-commutative spaces according to $\{\omega_1, \omega_2, \ldots, \omega_D\}$.

Figure 1 is, of course, valid in any number of dimensions, and we can construct corresponding transformations in a way analogous to the two-dimensional case.

After defining the Hamiltonian, we can also discuss the group of linear transformations $G(H) \subset Gl(2D, \mathbb{R})$ that keep the Hamiltonian invariant. For the non-commutative harmonic oscillator, this group is $O(2D)$. The degenerate energy levels for a given Hamiltonian are described by a set of orthogonal eigenstates transforming according to an irreducible representation of the dynamical symmetry group. The dynamical symmetry group $G(H, M)$ is the group of all transformations preserving both, M and the Hamiltonian, i.e., $G(H, M) = G(H) \cap G(M)$. For fixed Hamiltonian, the dynamical symmetry depends on M, so, by changing the parameters of the matrix M we can change $G(H, M)$ from $G_{\min}(H, M)$ to $G_{\max}(H, M)$. For the non-commutative harmonic oscillator, the minimal dynamical symmetry group is $[U(1)]^D$, and the maximal symmetry is $U(D)$. Note, however, that after fixing both the Hamiltonian and M, all systems connected to (H, M) by linear transformations will have dynamical symmetry groups isomorphic to each other.

Hence, different choices of M correspond to different dynamical symmetries. This can be viewed as a new mechanism of symmetry breaking with the origin in (phase)space structure. There are possible applications to bound states in atomic, nuclear and particle physics. From the symmetry-breaking effects in these systems one can, in principle, extract upper limits on the non-commutative parameters.

4 Harmonic oscillator: an example

In order to illustrate the general claims from the preceding sections, we choose a simple harmonic oscillator in two dimensions as an example. The $O(4)$ invariant Hamiltonian in this case is

$$H = \frac{1}{2}\sum_{i=1}^{4} U_i^2. \tag{18}$$

The constants $\hbar$, m and ω are absorbed in the phase-space variables. We parameterize the matrix M by four parameters:

$$M = \begin{pmatrix} 0 & 1 & \theta & \phi_1 \\ -1 & 0 & \phi_2 & B \\ -\theta & -\phi_2 & 0 & 1 \\ -\phi_1 & -B & -1 & 0 \end{pmatrix}. \tag{19}$$

The eigenvalues of the matrix iM are

$$\begin{aligned}\omega_{1,2} = &\frac{1}{2}\sqrt{(\theta-B)^2+(\phi_1+\phi_2)^2+4} \\ &\pm \frac{1}{2}\sqrt{(\theta+B)^2+(\phi_1-\phi_2)^2},\end{aligned} \tag{20}$$

and the spectrum of the Hamiltonian (18) is $E = \omega_1(n_1 + 1/2) + |\omega_2|(n_2+1/2)$ [9]; see (27) below. If the product of eigenvalues is positive, we are in phase I, and if negative in phase II. The frequency ω_1 is always positive, and ω_2 changes the sign in phase II. Duality relations between the two phases are obtained by demanding that the physical systems in both phases have the same energy spectrum. In the simple case $\phi_1 = \phi_2 = 0$, we have a one-to-one correspondence between (θ, B) and (θ', B'):

$$\begin{aligned}\theta &= \frac{1}{2}\left[\sqrt{(\theta'-B')^2+4}+\sqrt{(\theta'+B')^2-4}\,\right], \\ B &= \frac{1}{2}\left[\sqrt{(\theta'-B')^2+4}-\sqrt{(\theta'+B')^2-4}\,\right],\end{aligned} \tag{21}$$

and

$$\begin{aligned}\theta' &= \frac{1}{2}\left[\sqrt{(\theta-B)^2+4}+\sqrt{(\theta+B)^2-4}\,\right], \\ B' &= \frac{1}{2}\left[\sqrt{(\theta-B)^2+4}-\sqrt{(\theta+B)^2-4}\,\right].\end{aligned} \tag{22}$$

A comment is in order. Notice that the relations (21) and (22) are valid for $|\theta+B|>2$ and $|\theta'+B'|>2$, respectively. This is a sole consequence of the oversimplified parameterization $\phi_1=\phi_2=0$. For every point in parameter space there exists a dual point; we just have to allow for the most general parametrization of M. Finally, from $\omega_1\omega_2 = -\omega_1'\omega_2'$, we obtain

$$1-\theta B = \theta' B' - 1. \tag{23}$$

This condition is necessary but not sufficient in order to have the energy spectra in the two phases identical. A special case ($\theta=\theta'$) of this relation was obtained in [9], by considering the limit from the fuzzy sphere to the plane, for the Landau problem.

Although the systems depicted in Fig. 1 are physically distinct, the dynamical symmetry groups are all isomorphic to each other. At every point in Fig. 1 the generic symmetry ($\omega_1 \neq |\omega_2|$) is $U(1)\times U(1)$. We have only one quadratic symmetry generator, in addition to the Hamiltonian

$$\mathcal{G} = \sum_{i,j} C_{ij}U_iU_j, \quad [\mathcal{G}, H] = 0. \tag{24}$$

The matrix C is symmetric, commutes with M, $[C,M]=0$, and we can always choose Tr$C=0$. Then C^2 is proportional to the identity matrix. Namely, the C^0 matrix for the system (U^0, J_ω) is $C^0 \sim \mathrm{diag}(1,1-1,-1)$. Using the R transformation $U = RU^0$ we obtain $C = RC^0R^{\mathrm{T}}$ implying $C^2 \sim \mathbb{1}_{4\times 4}$. For the matrix M (19), the generator commuting with the Hamiltonian (18) is

$$\begin{aligned}\mathcal{G} = &\frac{1}{1-\theta B+\phi_1\phi_2}\Big\{(B+\theta)(X_1P_2-X_2P_1) \\ &-\frac{1}{4}(\theta^2-B^2+\phi_1^2-\phi_2^2)X_1^2 \\ &-\frac{1}{4}(\theta^2-B^2-\phi_1^2+\phi_2^2)X_2^2 \\ &+\frac{1}{4}(\theta^2-B^2+\phi_1^2-\phi_2^2)P_1^2 \\ &+\frac{1}{4}(\theta^2-B^2-\phi_1^2+\phi_2^2)P_2^2 \\ &-(\phi_1-\phi_2)(X_1X_2+P_1P_2)-(B\phi_1+\theta\phi_2)X_1P_1 \\ &-(B\phi_2+\theta\phi_1)X_2P_2\Big\}.\end{aligned} \tag{25}$$

One is tempted to call this generator the angular momentum, but this requires caution, as we have already discussed. For example, in the system $(H(RU^0), J_\omega)$ we cannot construct the angular momentum because $[E, J_\omega]\neq 0$. However, the symmetry generator for this system is

$$\mathcal{G}_0 = \frac{1}{2\sqrt{\omega_1|\omega_2|}}\left(X_1^{0\,2}+P_1^{0\,2}-X_2^{0\,2}-P_2^{0\,2}\right).$$

There are special points in parameter space of enhanced symmetry. In the special case $\omega_1=\omega_2$ (in phase I), we have the $U(2)$ symmetry group. In this case $\hbar_1 = \hbar_2 = 1$, $B=-\theta$ and $\phi_1=\phi_2=\phi$ and we can construct three generators of the dynamical symmetry satisfying the $SU(2)$ algebra $[L_i, L_j] = \mathrm{i}\varepsilon_{ijk}L_k,\ i,j,k=1,2,3$:

$$\begin{aligned}L_1 = &\frac{1}{1+\theta^2+\phi^2}\Big[X_1P_2-X_2P_1-\phi(X_1P_1+X_2P_2) \\ &-\frac{\theta}{2}(X_1^2+X_2^2-P_1^2-P_2^2)\Big], \\ L_2 = &\frac{1}{1+\theta^2+\phi^2}\Big[-P_1P_2-X_1X_2+\theta(X_1P_1-X_2P_2) \\ &+\frac{\phi}{2}(X_2^2-X_1^2+P_1^2-P_2^2)\Big], \\ L_3 = &\frac{1}{1+\theta^2+\phi^2}\Big[\frac{1}{2}(X_1^2-X_2^2+P_1^2-P_2^2) \\ &+\theta(X_1P_2+X_2P_1)-\phi(X_1X_2-P_1P_2)\Big].\end{aligned} \tag{26}$$

The dual point with $\omega_1 = -\omega_2$, with the $SU(2)$ symmetry in phase II, is obtained with $B = \theta, \phi_1 = -\phi_2 = \phi, \hbar_1 = -\hbar_2 = 1$. We wish to emphasize that the $SU(2)$ symmetry exists only for a special choice of parameters, and is not a dynamical symmetry of the Hamiltonian in the generic case (in contrast to the claims in [15]).

The transformations $\mathcal{L}, \mathcal{L}_0, S, D, R$ connecting different representations (see Fig. 1) of the harmonic oscillator on the non-commutative plane were discussed in the preceding section, and, partly, in the literature [9,14,15]. Using the matrix $\mathcal{L}_0 = RD$ we can transform the Hamiltonian (18) into an ordinary QM system:

$$\begin{aligned} H(U) = H(RDu^0) &= \frac{1}{2}\mathcal{L}^0_{ik}\mathcal{L}^0_{il}u^0_k u^0_l \\ &= \frac{\omega_1}{2}(u^{0\,2}_1 + u^{0\,2}_2) + \frac{|\omega_2|}{2}(u^{0\,2}_3 + u^{0\,2}_4). \end{aligned} \quad (27)$$

Next, we calculate the matrix elements of the observables starting form the ordinary harmonic oscillator observables:

$$\langle U_i \cdots U_k \rangle = \mathcal{L}^0_{ij_1} \cdots \mathcal{L}^0_{kj_k} \langle u^0_{j_1} \cdots u^0_{j_k} \rangle.$$

For quadratic observables in the ground state we use $\langle u^{0\,2}_i \rangle = 1/2$, $\langle u^0_1 u^0_2 \rangle = \langle u^0_3 u^0_4 \rangle = \mathrm{i}/2$, all others are zero. For the special case $\phi_1 = \phi_2 = 0$, we use the matrix R (16) to obtain

$$\begin{aligned} \langle X^2_1 \rangle = \langle X^2_2 \rangle &= \frac{1}{2}\left[\omega_1 \cos^2\varphi + |\omega_2| \sin^2\varphi\right], \\ \langle P^2_1 \rangle = \langle P^2_2 \rangle &= \frac{1}{2}\left[\omega_1 \sin^2\varphi + |\omega_2| \cos^2\varphi\right]. \end{aligned} \quad (28)$$

These expressions are universal, i.e., they are valid in both phases and at the critical point.

Here, we would like to comment on the uncertainty relations following from the commutation rules which define the theory. In the simple case $\phi_1 = \phi_2 = 0$, we have four non-trivial uncertainty relations $\Delta U_i \Delta U_j \geq |M_{ij}|/2$, i.e.,

$$\langle X^2_a \rangle \langle P^2_a \rangle \geq \frac{1}{4}, \; a = 1, 2, \quad (29)$$

$$\langle X^2_1 \rangle \langle X^2_2 \rangle \geq \frac{\theta^2}{4}, \; \langle P^2_1 \rangle \langle P^2_2 \rangle \geq \frac{B^2}{4}. \quad (30)$$

We calculate the left-hand side of the relations (29) and (30) in the ground state, using (28) and (17). In phase I we can saturate the first two relations (29) for $\theta = B$. In phase II we can saturate the other two relations (30) for any B and θ. At the critical point $\theta B = 1$ all four relations are saturated in the ground state. In the special case in phase I, $B = 0, \theta \neq 0$, none of the four uncertainty relations are saturated, in agreement with the theorem valid for quantum mechanics on the non-commutative plane with $B = 0$ [25]. This short analysis also indicates that physics in different phases is qualitatively different and depends crucially on M.

An especially interesting physical system is the Landau problem in the non-commutative plane, defined by $H = \mathbf{P}^2/2$ and the matrix M:

$$M = \begin{pmatrix} 0 & 1 & \theta & 0 \\ -1 & 0 & 0 & B \\ -\theta & 0 & 0 & 1 \\ 0 & -B & -1 & 0 \end{pmatrix}. \quad (31)$$

This problem can be treated as a non-commutative harmonic oscillator $H = \mathbf{P}^2/2 + \omega^2\mathbf{X}^2/2$, in the limit when $\omega \to 0$. We simply define $U_1 = \omega X_1$, $U_3 = \omega X_2$ to obtain a new matrix M_ω

$$M_\omega = \begin{pmatrix} 0 & \omega & \omega^2\theta & 0 \\ -\omega & 0 & 0 & B \\ -\omega^2\theta & 0 & 0 & \omega \\ 0 & -B & -\omega & 0 \end{pmatrix}, \quad (32)$$

with the determinant $\det M_\omega = \omega^4(1 - \theta B)^2$. We find the magnetic length (the minimum spatial extent of the wave-function in the ground state) in a universal form, valid in both phases and at the critical point:

$$\langle X^2_1 + X^2_2 \rangle = \langle r^2 \rangle = \frac{|1 - \theta B| + 1 + \omega^2\theta^2}{\sqrt{(\omega^2\theta - B)^2 + 4\omega^2}}. \quad (33)$$

In the limit $\omega \to 0$, the eigenvalues[1] of the matrix M_ω are $\omega_1 = |B|$, $\omega_2 = 0$ and the magnetic length is

$$\langle r^2 \rangle = \begin{cases} \frac{2-\theta B}{|B|}, \text{ if } \; \theta B < 1, \\ \theta', \quad \text{if } \; \theta' B' > 1. \end{cases} \quad (34)$$

For $|B| = B'$ these two expressions are the same if the duality relation (23) holds.

The above representation of the non-commutative Landau problem as a case of the non-commutative harmonic oscillator with $\omega \to 0$ can also be viewed as a non-commutative harmonic oscillator with $\tilde{\omega} \neq 0$, at the critical point $\tilde{\theta}\tilde{B} = 1$. The connection between the parameters is $\tilde{\omega}^2\tilde{\theta} + 1/\tilde{\theta} = B$. If we insist on having the same magnetic length in both pictures, we can fix $\tilde{\theta}$ and $\tilde{\omega}$.

However, these systems (the Landau problem with $\omega = 0$ and the harmonic oscillator with $\tilde{\omega} \neq 0$) are not physically equivalent. They have just the same energy spectrum and the same magnetic length if we choose so. A simple way to see this is to consider the uncertainty relations in phases I and II for the Landau problem, and at the critical point for the harmonic oscillator. Here we wish to emphasize once more that the only system having equality for both the spectrum of the Hamiltonian and the matrix of the commutators M are physically equivalent.

5 Conclusion

We have presented a unified approach to NCQM in terms of non-commutative coordinates and momenta in arbitrary dimensions and for arbitrary c-number commutation relations. We have considered all representations of

[1] In the "exotic" approach [21] $\omega^{\mathrm{ex}}_1 = \omega_1/\kappa = 1/\omega_2$, $\omega^{\mathrm{ex}}_2 = \omega_2/\kappa = 1/\omega_1$, and in the limit $\omega \to 0$ eigenvalues are $|\omega^{\mathrm{ex}}_1| \to \infty$, $\omega^{\mathrm{ex}}_2 \to 1/|B|$

NCQM connected by linear transformations from $Gl(2D, \mathbb{R})$ preserving the property that the commutation relations remain independent of the phase-space variables and keeping the energy spectrum of the system fixed. Among these, only the representations connected by transformations preserving the commutation relations are physically equivalent. We classify the non-commutative spaces according to the eigenvalues of the matrix iM, $\{\omega_1, \omega_2, \ldots, \omega_D\}$. The sign of the product of eigenvalues determines the phase, and the degeneracy among the $|\omega_i|$ determine the complete symmetry structure of phase space. Since orthogonal transformations keep the spectrum of the matrix iM fixed, they have been analyzed in detail. We have shown that for general M the angular momentum operator in the usual sense might not exist, and we have given the condition for its existence. An important result is that two physically distinct phases exist in arbitrary dimensions and that they are connected by discrete duality transformations.

Besides the symmetry structure of space, we have also discussed the dynamical symmetry of a physical system and proposed a new mechanism for symmetry breaking, originating from the phase-space structure.

In our approach to symmetries, there is no physical principle telling us what H and M we have to choose in terms of the non-commuting variables U. One way to test the idea of non-commutativity is to choose the Hamiltonian as in ordinary quantum mechanics, and to search for (tiny) symmetry-breaking effects induced by the phase-space structure M. The opposite way [16] is to fix the dynamical symmetry structure as in ordinary quantum mechanics. In the latter case, the differences should appear in the matrix elements of the observables and energy eigenstates. Of course, one can choose a combination of both approaches. Regardless of the approach, non-commutativity offers a new explanation of symmetry breaking, or change in probability amplitudes as a consequence of the phase-space (space-time) structure.

Our general approach enabled us to obtain new results even in the simplest case of the two-dimensional harmonic oscillator. We expect that we shall also obtain physically interesting results in the $D = 3$ case, currently under investigation.

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References

1. C.K. Zachos, Int. J. Mod. Phys. A **17**, 297 (2002); A.C. Hirshfeld, P. Henselder, Am. J. Phys. 70 (2002)
2. E. Witten, Nucl. Phys. B **460**, 33 (1996)
3. N. Seiberg, E. Witten, J. High Energy Phys. **09**, 032 (1999)
4. M.R. Douglas, N.A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001) and references therein
5. A. Connes, M.R. Douglas, A. Schwarz, J. High Energy Phys. **02**, 003 (1998)
6. L. Susskind, The Quantum Hall Fluid and Non-Commutative Chern Simons Theory, hep-th/0101029; A.P. Polychronakos, J. High Energy Phys. **04**, 011 (2001); L. Jonke, S. Meljanac, J. High Energy Phys. **01**, 008 (2002); Phys. Rev. B **66**, 205313 (2002)
7. J. Madore, S. Schraml, P. Schupp, J. Wess, Eur. Phys. J. C **16**, 161 (2000); X. Calmet, B. Jurco, P. Schupp, J. Wess, M. Wohlgenannt, Eur. Phys. J. C **23**, 363 (2002)
8. V.P. Nair, Phys. Lett. B **505**, 249 (2001)
9. V.P. Nair, A.P. Polychronakos, Phys. Lett. B **505**, 267 (2001)
10. A. Jellal, J. Phys. A **34**, 10159 (2001)
11. B. Morariu, A.P. Polychronakos, Nucl. Phys. B **610**, 531 (2001)
12. S. Bellucci, A. Nersessian, C. Sochichiu, Phys. Lett. B **522**, 345 (2001); S. Bellucci, A. Nersessian, Phys. Lett. B **542**, 295 (2002)
13. C. Acatrinei, J. High Energy Phys. **09**, 007 (2001)
14. A. Hatzinikitas, I. Smyrnakis, J. Math. Phys. **43**, 113 (2002)
15. A. Smailagic, E. Spallucci, Phys. Rev. D **65**, 107701 (2002)
16. O. Espinosa, P. Gaete, Symmetry in non-commutative quantum mechanics, hep-th/0206066
17. L. Mezincescu, Star Operation in Quantum Mechanics, hep-th/0007046
18. J. Gamboa, M. Loewe, F. Mendez, J.C. Rojas, Int. J. Mod. Phys. A **17**, 2555 (2002)
19. H. Falomir, J. Gamboa, M. Loewe, F. Mendez, J.C. Rojas, Phys. Rev. D **66**, 045018 (2002)
20. M. Chaichian, A. Demichev, P. Presnajder, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Lett. B **527**, 149 (2002)
21. C. Duval, P.A. Horváthy, Phys. Lett. B **479**, 284 (2000); P.A. Horváthy, Ann. Phys. **299**, 128 (2002)
22. R. Banerjee, Mod. Phys. Lett. A **17**, 631 (2002)
23. A. Deriglazov, Non-commutative quantum mechanics as a constrained system, hep-th/0112053; Phys. Lett. B **530**, 235 (2002)
24. D. McDuff, D. Salamon, Introduction to symplectic topology (Oxford Science Publications 1998)
25. K. Bolonek, P. Kosinski, Phys. Lett. B **547**, 51 (2002)